When first proposed in 1957, the BCS theory for superconductivity, which explained the quasi-totality of its thermodynamic and transport properties, was greeted with great circumspection, before it became the playground of particle physicists, who largely contributed to understand the deep physics behind this phenomenon. In the course of this undertaking, revolutionizing new concepts in physics were brought to light, such as (i) the "physical significance of the phase" of a quantum mechanical wave function whose role in force-transmitting gauge fields is to control the interaction between elementary particles in a current conserving manner (ii) "spontaneous symmetry breaking" and its related to it collective Nambu-Goldstone modes, which encode the basic symmetry properties of a quantum vacuum and the associated to them conserved quantities, (iii) The "Anderson-Higgs mechanism" and its associated to it, but so far experimentally unconfirmed "Higgs field", which provides the introduction of massive force transmitting gauge fields and ultimately the mass of elementary particles. These concepts were vital in consolidating the standard model for elementary particles and presented the final answer to what distinguishes a superconducting from a non-superconducting state: "electromagnetic gauge symmetry breaking", whatever the microscopic mechanism for that might be. We illustrate here how these concepts gradually emerged, once the basic features, which characterize superconductivity - the Meissner-Ochsenfeld magnetic field screening and the fundamental London equations explaining them in a phenomenological way - were established.

keywords: Superconductivity, electromagnetic gauge invariance breaking, Proca equation, Nambu-Goldstone modes, massive photons, Anderson-Higgs mechanism

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I. INTRODUCTION

Celebrating anniversaries of important scientific discoveries are occasions to reflect on the fundamental issues which were discussed at the time and which have since been largely forgotten, if not put deliberately aside in order to present popularized pedagogical descriptions in an easily accessible fashion to a broad scientific community. By doing this, the subtlety and profoundness of the arguments, which initially led to the comprehension of such discoveries are frequently lost. To remind oneself of the original intellectual effort necessary to grasp fundamentally new and basically unexpected physical phenomena, is important and not just for historical reasons. Commemorating the discovery of superconductivity by Kamerlingh Onnes in April 1911, one faces the problem of presenting a subject about which everything, which possibly could have been said, had been said. Yet, this enigmatic phenomenon of superconductivity has lost nothing of its original, still intriguing and to some extent poorly understood fundamental features. Our desperate struggle, for now a quarter of a century, to unravel the mysteries of the so-called high temperature cuprate superconductors witnesses that. I shall review here, the seemingly "never ending" hurdles which had and still have to be overcome in order to grasp the phenomenon of superconductivity and demonstrate how our illustrious predecessors in this process acquired a deep physical understanding of quantum mechanics and the subsequently emerging quantum field theory, which was instrumental for the achievements in particle physics throughout the second half of last century.

The present essay is based on a talk, which I presented in April 2011 to a broad scientific community, composed of solid state, nuclear and particle physicists, in a centennial celebration of the discovery of superconductivity, organized by the French Physical Society in Grenoble. The discussions, comments and questions, which this presentation sparked off have largely helped me to clarify many still nebulous points. I hope that with this review on superconductivity traversing the twenty's century, newcomers into the field might look at it as a challenging subject which still has many surprises up its sleeves.

The prime task of physics has always been and always will be, to reduce the diversity of physical phenomena, which we are confronted with by our experimental observations, to a strict minimum of basic laws of nature which might govern it. They represent conservation laws which control the dynamical processes of particles in their interaction with each other as well as in their decay processes. Such laws are formulated in terms of conserved quantities, which reflect fundamental symmetry properties of our universe. In classical Newtonian mechanics, space-time symmetries reflect the conservation of linear and angular momentum and energy. In classical electrodynamics they assure the conservation of charge through the invariance of the Maxwell's equations under a space-time correlated Lorentz transformation and a concomitant gauge transformation of the electromagnetic field tensor. Trying to comprehend our physical world along such lines, Galileo Galilei conjectured that the dynamics of our sub-lunar and supra-lunar world should be controlled by one and the same mechanism. Isaac Newton reconciled terrestrial motion with that of celestial one on a quite general level. Michael Faraday, discovering electromagnetic induction, paved the way to Maxwell's unification of electricity and magnetism. A turning point came to this approach when Einstein inverted this so-far applied principle by hypothesizing rather than by deriving general symmetry properties from experimental observations. From his proposition of a global and continuous space-time symmetry, he concluded that the laws of our physical world were controlled by special relativity, rather than classical mechanics.

Attempting to understand the physics on a subatomic level and to predict and classify the "elementary" particles: the Leptons (the electrons, the muons and the tau mesons, together with their corresponding neutrinos) and the six quarks, interacting with each other via force-transmitting gauge bosons (the massless photons and gluons and the massive $Z^0$ and $W^\pm$ bosons), quantum field theory introduced a set of new conservation laws. They complemented the classical laws, related to space-time symmetries in form of the CPT theorem and introduced the so-called "internal symmetries". The CPT theorem, involving charge conjugation, parity and time reversal transformations, requires that any combination of all three leaves any elementary particle as well as interaction invariant. The internal symmetries are at the origin of conservation laws which control the different quantum numbers, associated to the elementary particles and their force-transmitting gauge fields: The conservation of Barion and Lepton number, electric charge, hypercharge, Isospin, quark-colour and quark-flavour.

By incorporating those "internal symmetries" in a covariant Lagrangian formulation and subsequently searching for solutions which break these symmetries, enabled particle physicists to predict elementary particles together with their force transmitting excitations, which could be rendered visible in nuclear decay processes. The radioactive $\beta$ decay represented such an occasion for that: it breaks parity symmetry but not PC and conserves the Lepton and Barion number. Sheldon Glashow, Steven Weinberg and Abdus Salam subsequently showed that, as a consequence the electromagnetic forces (controlling the interaction between electrons) and the weak nuclear forces (involved in this parity breaking process), although being widely different in strength and range, represent just different manifestations of a common underlying electro-weak force. During the evolution of the universe, such different manifestations of a basically unique physics to start with, emerge through a hierarchy of spontaneous symmetry breaking processes, the outcome of which characterizes the state of our present day world.

The understanding of what makes a metal to become a superconductor and not merely a perfect resistance-less
conductor, was crucial in arriving at the level of understanding such basic principles and laws of nature. Fundamentally new concepts in physics were unearthed in this process. They acquired their profound physical implications and full understanding really only once Bardeen, Cooper and Schrieffer in their BCS theory\textsuperscript{11} had understood and described the quasi totality of the basic thermal and transport properties of superconductors. Their microscopic approach focused on the superconducting gap in the single-particle spectrum of the electrons, considering the gap to play the role, at that time, of an order parameter. The real robust features of the superconducting state, which substantially revised our concepts in physics at large, however became apparent from its macroscopic manifestations: the Meissner-Ochsenfeld effect\textsuperscript{13}, the flux quantization\textsuperscript{14,15} and the Josephson effect\textsuperscript{16} in conjunction with their description in terms of the phenomenological Ginzburg-Landau\textsuperscript{17} theory. This theory derived its input from the Meissner-Ochsenfeld effect and Fritz London’s early conjecture\textsuperscript{18} of a truly macroscopic wave function describing the current carrying superconducting state, tantamount to a macroscopic diamagnetic molecule. The year after that, when superfluidity of \textsuperscript{4}He was discovered\textsuperscript{19,20} and the connection between superconductivity and superfluidity became apparent, Fritz London’s picture for superconductivity, was determinant for pinning down the robust features behind this phenomenon: The first one was its connection to a Bose-Einstein condensation\textsuperscript{21,22} which for the first time acquired a true physical significance. The second and even more important feature, as far as superconductivity is concerned, was its intrinsic perfect diamagnetism. It is related to persistent super-currents, driven by a self-sustaining mechanism which relies on a qualitative modification of the classical Maxwell equations of electromagnetism. As Fritz and Heinz London showed, these equations had to be complemented by what has been termed the London equations\textsuperscript{23}. The implications of that however go far beyond the explanation of the Meissner-Ochsenfeld screening of a magnetic supercurrentetic field from the interior of a superconductor. It led to fundamentally new concepts, which were determinant for our understanding of quantum mechanics and its evolution into quantum field theory and particle physics, such as:

i) The phase of matter waves - gauge bosons and gauge fields.

In the early days of quantum mechanics, in the beginning of the last century, the phase of de Broglie matter waves was considered at best as a ”redundant degree of freedom” - an indeterminable gauge of the wave function of no particular physical significance. Yet, a deep physical meaning could not be denied for a similarly indeterminable gauge of the four-component electromagnetic vector-scalar field of the Maxwell equations. The invariance of the Maxwell equations under gauge transformations are rooted in an underlying conserved quantity: the electric charge of the particles. When charge carriers interact with such an electromagnetic field, the Lagrangian describing this interaction has to be gauge invariant. The albeit arbitrary phase of the quantum mechanical wave functions of the charged particles then becomes irrevocably tied to the gauge of the vector-scalar components of the electromagnetic field in which they are embedded. The electromagnetic properties of superconductors, permitted us to visualize the physical effect of gauge transformations, relating the dynamics of massive charge carriers (the electrons) to that of massless gauge bosons, playing the role of the local force-transmitting excitations which accompany them.

It is generally attributed to Wolfgang Pauli\textsuperscript{24} for having introduced the local gauge theory approach into particle physics. The local gauge in quantum field theories - the Yang-Mills gauge theories\textsuperscript{25} - assures the conservation laws controlling the dynamics of elementary particles together with their force transmitting gauge bosons. The basic idea behind that was the same as that which Emmy Noether had laid down before in her ”Noether’s theorem”\textsuperscript{26,27} (For a modern discussion of it, see ref. [28]). It states that the conservation of quantities, such as momentum, angular momentum and energy arises from basic symmetries of our classical physical world: homogeneity in space and time and isotropy. From that alone follows that the dynamics in such systems must be invariant with respect to continuous global transformations of the space-time and rotational coordinates.

The ”quantum field theory” extended these so-called Poincaré symmetries of classical space-time translational invariance to ”internal symmetries”, characterizing elementary particles and their associated gauge bosons. The latter present the quantized excitations of the force transmitting fields, similar to those of photons in the electromagnetic field which accompany the electrons. Gauge field theories, culminating in the standard model, assume as an outset a highly symmetric quantum vacuum, composed of a multitude of different degrees of freedom. Following the BIG BANG, the universe evolved by expanding and thereby cooled down. In this course of events, a hierarchy of symmetry breaking processes occurred which ultimately led to our presently observed state of forces, widely differing in strength and range: the electromagnetic, the weak and the strong nuclear forces. The gravitational force remains outside of such a scheme. Integrating it into it would require to treat general relativity (which considers the mass as a localized in space object) and quantum mechanics (which considers it as a spatially extended wave packet), on the same footing. Presently, it is not known how this could be achieved.

ii) Spontaneous symmetry breaking - Nambu-Goldstone modes.

Spontaneous symmetry breaking (SSB) implies that a physical system with an internal global continuous symmetry, resulting in a certain conserved quantity, has a ground state in which this symmetry is broken without any external perturbation causing it. SSB was conjectured by Phil Anderson\textsuperscript{29} in connection with the breaking of the continuous rotational symmetry of the phase in the superconducting ground state. The infinite multiplicity of its degenerate ground state leads to the emergence of long wave length massless spin-0 scalar boson collective excitations, known
as Nambu-Goldstone modes (under the condition that the interactions in such a system are short ranged), which reestablish the original global symmetry in a dynamical fashion. Nambu showed that these modes contribute to the charge current in a way, without which it would not be conserved. We shall illustrate in Sec. VI how this comes about, on the basis of a concrete example, that of a charged superfluid Bose liquid.

The Nambu-Goldstone modes distinguish themselves from massless excitations in systems with unbroken or weakly broken symmetries in the sense, that they encode the structure of the originally symmetry-unbroken situation. An example for that is the fluctuating microwave background, which emerged after the BIG BANG in form of a SSB state of the "inflationary" fully symmetric behavior of the our universe as it existed before it.

It is generally accepted to credit Nambu as the most instrumental physicist for having transposed the concepts of the SSB in superconductivity to Lorentz covariant gauge field theories, as recalled by one of his collaborators, Jona Lasinio as well as by Steven Weinberg. Its outcome was the rigorous formulation of the "Goldstone theorem", which stated that upon breaking an exact global continuous symmetry, necessarily leads to the emergence of massless Nambu-Goldstone bosons. This theorem played a considerable role in the construction of the standard model for elementary particles, whose major aim was to bring order and classification into the flood of experimental discoveries of potentially elementary particles and to reduce the fundamental laws of physics acting between them to a strict minimum. Its first success was the unification of the electromagnetic and the weak nuclear forces by Sheldon Glashow, Steven Weinberg and Abdus Salam, to which we have alluded above. But this necessitated having to overcome yet another and major obstacle: The force-transmitting gauge fields involved in the β decay, which was at the basis of this unification claim, experimentally turned out to have mass. This was in flagrant dis-accord with the concept of the intrinsic masslessness of gauge bosons. It implied the violation of gauge invariance and hence called into question the current conservation. Once more, the phenomenon of superconductivity, played a decisive role in resolving this riddle, through the "Anders-Higgs mechanism".

iii) The emergence of massive gauge bosons - The Anderson-Higgs mechanism.

The experiment, which pinned down once and for all the singular behavior of superconductors, was the screening of an externally applied magnetic field from the interior of a persistent current-carrying superconducting hollow Pb sphere, observed by Walther Meissner and Robert Ochsenfeld in 1933. More then 20 years had passed since the discovery of superconductivity, when this observation finally led to its ultimate understanding. It required a qualitative modification of the electromagnetic field which accompanies such a persistent current as soon as the superconducting state sets in below a critical temperature $T_c$. This current is self-sustained, persistent and geometrically restricted to a thin layer near the surface of this sphere, which prevents any electromagnetic field to penetrate in its interior deeper than the Meissner-Ochsenfeld penetration depth. In 1935 Fritz and Heinz London showed that in order to describe such a situation, the local electromagnetic field, which accompanies this super-current, materializes if the current-density source terms of the Maxwell equations are self-determined. They have to be related back to those electromagnetic fields via two phenomenological equations, the so-called London equations. Effectively, this amounts to transform the Maxwell equations into Proca equations, which describe electric and magnetic fields which have a mass, inversely proportional to the penetration depth.

On a microscopic level this requires turning massless transverse photons of the electromagnetic field in vacuum, i.e., outside the superconductor, into massive excitations when they enter the superconducting material and interact with the circulating current. The conservation of this current is encoded in the characteristic properties of the Nambu-Goldstone mode of the spontaneously broken symmetry of the condensed state of the superconductor. But coupling this current to an electromagnetic field, locally breaks the gauge invariance of that latter. In order to assure the conservation of this current, a self-sustaining feedback effect between the current carried by the charge carriers and that carried by the accompanying electromagnetic vector field, establishes itself. This goes at the expense of the freely propagating features of the photons which accompany this current locally and which in this process are turned into massive excitations. This process has been termed the Anderson-Higgs mechanism. Its generalization to relativistic gauge theories was done by Higgs and independently by Englert and Brout and Guralnik, Hagen and Kibble. It made it possible to circumvent the intrinsic masslessness of the gauge fields when they are associated via a local gauge transformation to a massless Nambu-Goldstone mode. This makes this mode disappear as a physical particle, but resurrects it in form of a helicity-0 spin-1 vector boson with mass - presenting in a more complex scenario than presented here, the actively searched after Higgs Boson.

These concepts evolved gradually as the understanding of superconductivity progressed. The reason, why it has taken such a long time for that is rooted in our traditional reductionist approach to physical problems. The superconducting state is caused by electromagnetic gauge symmetry breaking, independent on any specific microscopic scenario. A concept which was difficult to accept. One should remember that up to and even still slightly after the publication of the BSC theory in 1957, it was tabu to consider that the gauge (the phase) of a matter wave could possibly have any physical role to play in superconductivity. The proposition that its "redundant" character could be lost by acquiring a definite value, which leads to gauge symmetry breaking, took even longer to be accepted. To show how this break-through of our thinking evolved throughout the last century, let me highlight below the
main milestones in this process, illustrating the thorny step by step advances in understanding the phenomenon of superconductivity, which has taken half a century after its discovery.

In order to make the meaning of the superconducting state clear, I shall in Sec. II illustrate its differences with respect to an ideal conductor. The main characteristics of a superconductor - the superconductor gap and the Meissner-Ochsenfeld magnetic field screening - will be discussed in Sec III. Combining the Maxwell equations with the phenomenological London equations in Sec IV, we shall highlight its effect on the introduction of a longitudinal mode in the electromagnetic excitations. They are forbidden in vacuum because of the Lorentz condition, but which becomes redundant for massive electromagnetic field equations. The absorption of the Nambu-Goldstone mode in a SSB state of the superconductor and its reappearance in form of a massive helicity-0 spin-1 vector boson field will be discussed in terms of the Ginzburg-Landau macroscopic formulation of superconductivity in Sec V. Finally, in Sec VI we shall treat this feature on the basis of a microscopic generic model for superconductivity in terms of a charged Bose gas in a manifestly gauge invariant manner and derive generalized London equations for spatial and temporarily varying electromagnetic source fields, which leads us up to the Proca equations for the electromagnetic fields accompanying and sustaining the super-current. In the final section VII we conjecture, how the fundamental features as "quantum protection" whose dynamical stability are assured by such concepts which originated from our understanding the BCS superconductivity might have to be complemented and integrated into a wider scheme, when addressing the physics of the high $T_c$ cuprate superconductors. We advance some preliminary ideas based on the concept of "emerging phenomena" whose dynamical stability are assured by such features as "quantum protection".

II. THE PERFECT CONDUCTOR AND THE MYTH OF THE FROZEN-IN MAGNETIC FIELD, 1911-1933

In 1908 Kamerlingh Onnes succeeded in liquefying gaseous $^4$He below 4.22 K. This earned him the 1913 Nobel prize and gave him a worldwide monopoly to explore physical phenomena at such low temperatures. It was in this way that superconductivity was discovered - without that anybody would have foreseen it - in 1911. In this experiment a magnetic field was applied traversing a ring of frozen Mercury. The temperature then was reduced to below some critical temperature $T_c$, in this case 4.15 K. When subsequently the external field was shut off, a persistent resistance-less current (super-current) manifested itself in the ring. It was evidenced in form of a magnetic flux threading through the ring, which was equal in strength to that of the field initially applied to it. It led to the long standing official doctrine that superconductivity was induced by the externally applied field having gotten frozen into the sample before cooling it down to below $T_c$. This interpretation was supported by considering superconductors as resistance-less "perfect conductors", as illustrated by Schoenberg, following an earlier study by Becker, Heller and Sauter. The argument goes as follows.

In a "perfect conductor", a super-current $J_s = n_s q_s \mathbf{x}$, with $\mathbf{x}$ representing its velocity, is accelerated in an applied local electric field $\mathbf{E}$ according to:

$$\partial_t J_s = \frac{n_s q_s^2}{m_s} \mathbf{E},$$

where $n_s$, $m_s$ and $q_s$ are the effective density, mass and charge of the particles making up such a super-current. Using Ampere’s and Faraday’s laws: $\nabla \times \mathbf{B} = J_s$ and $\nabla \times \mathbf{E} = -\frac{1}{c} \partial_t \mathbf{B}$, with $\mathbf{B}$ and $\mathbf{E}$ denoting the local magnetic and electric fields, we have

$$\frac{1}{c} \partial_t \nabla \times \mathbf{B} = \lambda_L^{-2} \mathbf{E} \implies \nabla^2 \mathbf{E} = \lambda_L^{-2} \mathbf{E},$$

with $\lambda_L = \sqrt{m_s c / n_s q_s^2} = \sqrt{1 / n_s r_s}$. $r_s = q_s^2 / m_s c$ plays the role of the radius of the charge carriers with charge and mass given by $q_s$ and $m_s$.

Requiring this super-current to be homogeneous ($\nabla \mathbf{E} \equiv 0$), it follows from Eq. [2] that it should flow without any voltage difference, i.e., $\mathbf{E} \equiv 0$ everywhere in the sample. Yet, considering that such a super-current is triggered by applying a magnetic field, things become bothering.

Taking the curl of Eq. [1] and using Faraday’s law we have:

$$\partial_t \left[ \nabla \times J_s + \frac{n_s q_s^2}{m_s c} \mathbf{B} \right] = 0.$$  

Eliminating subsequently $\mathbf{J}_s$ via Ampere’s law and taking into account that $\nabla \mathbf{B} \equiv 0$ it follows that:

$$\partial_t \left[ \nabla^2 - \frac{1}{\lambda_L^2} \right] \mathbf{B} \equiv 0,$$

with $\lambda_L = \sqrt{m_s c / n_s q_s^2} = \sqrt{1 / n_s r_s}$. $r_s = q_s^2 / m_s c$ plays the role of the radius of the charge carriers with charge and mass given by $q_s$ and $m_s$.

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Solving this equation for a half-infinite system with its surface orthogonal to the z-direction and for an externally applied field $B_0 = [B_0^x, 0, 0]$, polarized parallel to this surface, one has

$$\partial_t B^x(z,t) = \partial_x B_0^y(t)\exp(-z/\lambda_L).$$

(5)

This implies that any temporal variation of the externally applied field, like switching it on or off, will result in a relaxation of this perturbation to the system, which will have no effect on the flux in the sample, deeper than a distance $\lambda_L$, as measured from the surface. In other words, such a perturbation has no effect on any pre-existing magnetic flux inside the sample - a situation, which has unphysical consequences.

When a magnetic field is switched on in a perfect conducting ring in the resistanceless state below a certain temperature $T_c$, it "kicks off" a circulating super-current, following Lenz’s law and prevents in this way the initially applied magnetic field to penetrate into the bulk of the sample. When subsequently the external field is shut off, the circulating current stops and with it, its accompanying magnetic flux in the sample. The system returns to its initial state with no current flowing.

The situation is qualitatively different, when applying the magnetic field above $T_c$ in the resistive state of a "perfect conductor". The magnetic flux then traverses the bulk of the sample. Upon lowering the temperature to below $T_c$, the flux line distribution in the bulk remains unchanged and no super-current is being set off. In order to generate a super-current to flow, one must switch off the externally applied field. Following the relation, Eq. 5, which controls the relaxation of such a magnetic flux, its distribution inside the material will tend to that which it was, when the system was first exposed to a magnetic field above $T_c$. Hence the conclusion of a frozen-in magnetic field driving the super-current.

The resistanceless state thus depends on the order in which this state is approached: either by first lowering the temperature and then switching on the field or the inverse. It means that the superconducting state would not be a truly thermodynamically stable state. This highly unsatisfactory conclusion fortunately turned out to be wrong. But it has taken more than twenty years after the discovery of superconductivity that this was finally recognized, thanks to the experimental findings in 1933 by Meissner and Ochsenfeld, which stipulated that: (i) The field inside a superconductor is identically zero. The super-current is self-sustained by the field which it creates via Ampere’s law and screens any externally applied field. (ii) The order, in which the temperature and the magnetic field is varied in inducing the super-current carrying state and its associated characteristic screening features, is irrelevant.

This finding by Meissner and Ochsenfeld established without ambiguity the true thermodynamically stable state nature for superconductivity and its second order phase transition into a non-superconducting metallic phase. A superconductor hence proved to be more than simply a "perfect conductor". The observation of the "frozen-in magnetic field" in those early years could have been due a variety of different unfortunate circumstances: the shape of the sample, magnetic supercooling and flux trapping as the magnetic field is switched off.

III. THE TWO FUNDAMENTAL PROPERTIES OF SUPERCONDUCTORS: THE MACROSCOPIC QUANTUM PHASE COHERENCE AND THE SINGLE PARTICLE GAP

By 1928 the advances in Quantum Mechanics had sufficiently progressed to provide one with a theoretical description of de Broglie matter waves for the electrons in Arnold Sommerfeld’s and Felix Bloch’s theory of metals. By 1933, Ehrenfest provided us with the concept of second order phase transitions, which evolved into Lev Landau’s concept of an order parameter. Curiously, one of the most relevant concepts for describing the superconducting state, i.e., its coherent macroscopic quantum state, proposed by Erwin Schrödinger in 1926 remained totally unnoticed in the solid state community. Schrödinger’s proposition captured the classical features of a quantum system, such as to satisfy optimally the correspondence principle with a minimal Heisenberg quantum uncertainty. The concept of macroscopic quantum coherence resurfaced only after the observation of the total loss of viscosity in the flow of liquid $^4$He through pores and in general any obstacle, below a certain temperature $T_\lambda = 2.18$ K. The flow being controlled exclusively by its inertia, led Kapitza to call this feature: superfluidity. The observed specific heat cusp at the transition into this state gave it the name "lambda transition". It strongly resembled that expected for the specific heat of a free Bose gas, undergoing a condensation of the majority of bosonic $^4$He atoms into a single quantum state. Its transition temperature $T_{BEC} = (2\pi\hbar^2/1.8m_nk_B)n^{2/3}$, with $m_n$ given by the mass of the $^4$He atoms and $n \approx 10^{22}/cm^3$ being their density, comes surprisingly close to the observed value of $T_\lambda$. The frictionless flow of superfluid $^4$He in a tube without any difference in pressure on the two ends, is the analog of the resistanceless conductivity, i.e., the persistent flow of an electric current in a superconductor in the limit where the driving electric field goes to zero. Driving an electric super-current by applying a magnetic field in the Meissner-Ochsenfeld experiment found its analog much later in the rotating bucket experiment in $^4$He.

Fritz London and independently Tisza recognized that superfluid Helium and superconducting metals must contain basically the same underlying physics, inherent to Bose quantum liquids. Their ground state therefore should
be describable by a single macroscopic matter wave function. It implied generalizing Schroedinger’s phonon coherent state, accounting for the macroscopic features of a quantum harmonic oscillator, to that of an equivalent coherent state of massive bosonic $^4$He atoms. This concept ultimately led to the highly successful phenomenological Ginzburg-Landau theory of superconductivity\textsuperscript{17}. Penrose and Onsager\textsuperscript{56} consolidated this concept by attributing to such a macroscopic wave function the intrinsic property of “long range off-diagonal order”, which, according to C. N. Yang\textsuperscript{55} encodes the concept of ”spontaneous gauge symmetry breaking”.

The steady state of a persistent homogeneous super-current, circulating in a ring, which follows from such a macroscopic wave function for a condensate, suggested to Fritz London\textsuperscript{15} that any particle making up the super-current must be correlated to any other one, independent on the spatial difference between two. This pointed to electron pairing and effective charge carriers having charge $2e$. Fritz London moreover conjectured that such a current should behave similar to that of a rotating macroscopic diamagnetic molecule under the influence of an applied magnetic field. Since the quantum mechanical spatial correlations between any two particles in the current on a ring could be considered as independent on their relative distance, it implied that the conjugate momentum of it had to be fixed within the limits of Heisenberg’s uncertainty principle. Fritz London conjectured from that, that the rigidity of the macroscopic wave function of a super-current on a ring derived from pairing correlations in reciprocal rather than real space, limited to the momenta of the particles which could participate in such a process, i.e., those close to the Fermi surface since all others are blocked by the Pauli exclusion principle. This insight was instrumental for John Bardeen and his collaborators to construct ultimately the macroscopic coherent quantum state capable of the macroscopic wave function of a super-current on a ring derived from pairing correlations in reciprocal rather than real space, limited to the momenta of the particles which could participate in such a process, i.e., those close to the Fermi surface since all others are blocked by the Pauli exclusion principle. This insight was instrumental for John Bardeen and his collaborators to construct ultimately their macroscopic coherent quantum state capable of describing dynamical pairing of electrons rather than bound pairs, which would mimic the superfluidity of bosonic $^4$He atoms. That latter scenario was followed by Schafroth and his collaborators Blatt and Buttler\textsuperscript{56}, who tried to relate superconductivity to a Bose Einstein condensation of diamagnetic bound electron pairs. What eventually decided in favor of John Bardeen’s approach, was the experimentally observed gap in the electronic single-particle spectrum. Its difference with respect to an insulating gap is encoded in the specific structure of the macroscopic quantum state for the BCS wave function with (i) its coherence factors describing the intrinsically Fermionic system and (ii) its single particle excitations, derived by Bogoliubov\textsuperscript{42} and independently Valatin\textsuperscript{28} which are quantum superpositions of negatively charged electrons and positively charged holes.

Keesom’s and Kok’s early measurement of the specific heat\textsuperscript{59} of a superconducting tin sample showed a jump in the specific heat at $T_c$, which indicated the opening of a gap in the electron spectrum as one entered the superconducting phase. A definite proof for it came later in experiments on the change-over in the thermal conductivity\textsuperscript{59} upon entering the superconducting state with decreasing temperature. Since a super-current does not transport heat, this experiment permitted to follow the variation with temperature of this gap in the electron spectrum. It led John Bardeen\textsuperscript{61} to conjecture that the gap should play the role of an order parameter. Immediately after the publication of the microscopic BCS theory\textsuperscript{11} which aimed to account for the thermodynamic and transport properties of superconductors, this conjecture was elaborated by Gorkov.\textsuperscript{58} He succeeded to make the connection with the phenomenological Ginzburg-Landau theory\textsuperscript{47} and its complex order parameter, given by the macroscopic condensate wave function, whose modulus represents the value of the gap, but whose phase controls the onset of superconductivity.

IV. THE MEISSNER-OCHSENFELD EFFECT AND THE LONDON EQUATIONS AND HOW THEY INFLUENCED THE ELECTROMAGNETIC PROPERTIES

In interpreting the Meissner-Ochsenfeld experiment, Fritz and Heinz London\textsuperscript{24} stipulated that:

I) The friction-less conduction of super-carriers with some effective charge $q_s$, mass $m_s$ and charge/mass density $n_s$ manifests itself in the acceleration of a charge-current which, in the limit $E \to 0$ is described by the ”first London equation”, which is identical to Eq. \[( \frac{\partial}{\partial t} \mathbf{J}_s = \frac{n_s q_s^2}{m_s} \mathbf{E}. \]

II) The exclusively transverse component of such a steady state super-current $\mathbf{J}_s$, i.e., $\nabla \times \mathbf{J}_s \neq 0$, prevents any magnetic flux from entering the bulk of a superconducting material beyond a certain penetration depth $\lambda_L$ according to the ”second London equation”,

\[ \nabla \times \mathbf{J}_s = -\frac{n_s q_s^2}{m_s c} \mathbf{B}. \]

The effect of these two premises, although not spelled out at that time, is that the steady state of the super-current and its associated to it local electromagnetic field, as we understand it now, sustain each other mutually in the
sense that: (i) When a magnetic field is applied to the material above the critical temperature $T_c$, a super-current is spontaneously put into circulation as soon as the temperature is reduced to below $T_c$. Shutting off this external field, the super-current disappears. (ii) A material which has already been cooled to below $T_c$, exhibits a super-current as soon as an external magnetic field is applied to it. The super-current, accompanied by its electromagnetic screening field, persists until the external field is shut off.

As we have seen in Sec. II, a superconducting state, triggered by an external magnetic field below $T_c$, can adequately be accounted for its frictionless conduction in terms of a "perfect conductor", described by the "first London equation", Eq. 6. It however can neither explain the superconducting state in a field-cooled experiment nor the screening of the magnetic flux from the interior of the material. In order to remedy this situation, Fritz and Heinz London conjectured in 1935 that Eq. 3 for a "perfect conductor" should be replaced by the more stringent relation, expressed in their "second London equation", Eq. 7. Eq. 9 describing the "perfect conductor" then gets replaced by

$$\nabla^2 - \frac{1}{\lambda_L^2} B = 0$$

(8)

in the case of a superconductor. If one requires the magnetic flux to be homogeneous in the interior of the superconductor, it follows from Eq. 9 that $B$ must be identically zero in the bulk of the material beyond a certain penetration depth $\lambda_L$. Fritz and Heinz London conjectured from this that charged particles in the superconducting and in the normal state interact differently with an electromagnetic field. As we shall see below, this is related to the breaking of the local electromagnetic gauge invariance in the superconducting state.

Felix Bloch suggested that the "second London equation" results from the fact that the charged particles (making up the super-current) in the presence of a magnetic field $B(x) = \nabla \times A(x)$ should be described by the canonical momenta $p = m_s x + (q_s/c) A(x)$, rather than by the usual simple kinetic contribution $m_s x$, where $A(x)$ denotes the local vector potential. In the ground state, the total canonical momentum of the ensemble of charge carriers has to be equal to zero: $\langle 0| p |0 \rangle = 0$

(9)

from which follows the "second London equation". Eq. 9 visibly violates electromagnetic gauge invariance ($A(x) \rightarrow A(x) + \nabla \Lambda(x)$) and that therefore charged particles in the superconducting state, contrary to its normal state, defy the conservation of the charge-current. It took several years after the publication of the BCS paper in 1957 that this problem was remedied. As Nambu showed, the kinetic part of the charge current has to be complemented by a contribution $J_{coll} = \nabla f(x, t)$, which arises from collective modes, described by $f(x, t)$ and satisfying the wave equation

$$\left( \nabla^2 - \frac{1}{\alpha^2} \partial_t^2 \right) f(x, t) = -2 \Delta \Psi^\dagger \tau_2 \Psi(x, t).$$

(10)

$\Delta$ denotes the gap value, $\Psi(x, t)$ the electron-hole spinor wave function and $\alpha$ the plasma frequency of the free electron gas. This puts back Felix Bloch’s Ansatz, Eq. 9 into a gauge invariant form. How this happens in detail will be addressed in sections V and VI below.

For the moment let us pursue the classical phenomenological description of superconductivity, based on the London equations and illustrate how the electromagnetic field is modified, when it enters the superconductor and couples to its circulating currents. In a superconducting metal, the total current is composed of two contributions: $J = J_s + J_n$, where $J_n = \sigma E$ denotes that part of it which flows with a finite resistivity $1/\sigma$ and $J_s$ represents the super-current, which above $T_c$ is identically zero.

Complementing Maxwell equation in vacuum

$$\nabla \times B = J + \frac{1}{c} \partial_t E, \quad \nabla \times E = -\frac{1}{c} \partial_t B, \quad \nabla B = 0, \quad \nabla E = \rho,$$

(11)

whose corresponding wave equations for the electric and magnetic fields are given by

$$\left( -\nabla^2 + \frac{1}{c^2} \partial_t^2 \right) \begin{bmatrix} E \\ B \end{bmatrix} = \frac{4\pi}{c} \begin{bmatrix} \frac{-1}{c} \partial_t J - \nabla \rho \\ \nabla \times J \end{bmatrix}$$

(12)

with the London equations, Eqs. 6 and 7, we obtain wave equations

$$\left( -\nabla^2 + \frac{1}{c^2} \partial_t^2 + \frac{m_0^2 c^2}{\hbar^2} + \frac{1}{c} \sigma \partial_t \right) \{ B, E, J \} = 0,$$

(13)
not just for the usual transverse excitations of the electric and magnetic fields E and B but also for the longitudinal excitations associated to the current J. In order to derive this result we have put the charge source terms in the Maxwell equations Eqs. 11 equal to zero. In the Meissner-Ochsenfeld experimental set up, the sole external source initiating the circulation of a current comes from a magnetic field, i.e., from the transverse component of the vector potential and not from any distribution of localized charge carriers which would produce such an electric field source term.

All of these modes are massive, with a rest-mass \( m_0 \) given by \( m_0^2 c^2 / h^2 = \omega_q^2 / m_0 c \). These equations present a generalization of the Maxwell equations to finite mass photons and were initially conjectured by Proca on general field theoretical grounds in view of generalizing the Klein-Gordon equation for massive spin-0 scalar Boson fields to fields describing massive spin-1 vector Boson fields.

Let us now highlight the difference of the electromagnetic properties between the normal non-superconducting materials, described by the standard Maxwell wave equations Eqs. 12 and the superconducting materials, described by the Proca equations, Eqs. 13.

We begin with the non-superconducting case for which the standard Maxwell equations Eqs. 11 apply and rephrase these equations in their relativistically covariant form

\[
\nabla^2 \phi(x, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \phi(x, t) = - \frac{1}{e} \frac{\partial}{\partial t} \left( \nabla A(x, t) + \frac{1}{c} \frac{\partial}{\partial t} \phi(x, t) \right) - \rho(x, t)
\]

\[
\nabla^2 A(x, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} A(x, t) = \nabla \left( \nabla \phi(x, t) + \frac{1}{c} \frac{\partial}{\partial t} \phi(x, t) \right) - J(x, t),
\]

after having cast the physical electric and magnetic fields into a gauge invariant form by introducing the scalar and vector fields \([\phi(x, t), A(x, t)]\) with \( B(x, t) = \nabla \times A(x, t) \) and \( E(x, t) = -\nabla \phi(x, t) - \frac{1}{c} \frac{\partial}{\partial t} A(x, t) \). Eqs. 14 present historically the first example of a unification of different forces: the electric and magnetic forces. Their invariance under the electromagnetic gauge transformations \([\phi(x, t), A(x, t)] \rightarrow [\phi'(x, t) = \phi(x, t) - \frac{1}{c} \frac{\partial}{\partial t} \Lambda(x, t), A'(x, t) = A(x, t) + \nabla \Lambda(x, t)]\) \( \Lambda(x, t) \) being an arbitrary function of \( x \) and \( t \) results from an underlying symmetry of electromagnetism, which is associated to the conservation of charge.

In order to demonstrate that, let us operate with \( \frac{1}{c} \frac{\partial}{\partial t} \) onto the second of these two equations, Eqs. 13 and with \( \nabla \) onto the second one. When subsequently summing the two equations, we obtain the continuity equation \( \frac{1}{c} \frac{\partial}{\partial t} \rho(x, t) + \nabla J(x, t) = 0 \), which asserts that the Maxwell equations are charge-current conserving and that the symmetry associated to this conservation law is that of electromagnetic gauge invariance. Since the Maxwell equations are also invariant under the space-time Lorentz transformations, \([x', y', z' = (z - vt)/\sqrt{1 - (v/c)^2}, t' = (t - (vz/c^2))/\sqrt{1 - (v/c)^2}]\), it requires (see for details Ref. 63) that the charge-current conservation, i.e., the continuity equation, must be invariant under Lorentz transformations. This leads to the so-called "Lorentz condition"

\[
\nabla A(x, t) + \frac{1}{c} \frac{\partial}{\partial t} \phi(x, t) = 0,
\]

which correlates \( \phi(x, t) \) and \( A(x, t) \) and renders the Maxwell equations fully symmetric

\[
\nabla^2 \phi(x, t) + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \phi(x, t) = - \rho(x, t)
\]

\[
\nabla^2 A(x, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} A(x, t) = - J(x, t).
\]

It expresses the fact that the charge density and the current density together with their associated scalar and vector fields are merely different manifestations of a single and unique underlying physics.

The wave character of Eqs. 16 is contained in their homogeneous solutions, which can be parametrized by \( \phi(x, t) = e_\phi e^{-i(\mathbf{q} \cdot \mathbf{x} - c t)} \) and \( A(x, t) = e_\mathbf{A} e^{-i(\mathbf{q} \cdot \mathbf{x} - c t)} \). \( e_\phi \) and \( e_\mathbf{A} \) present the components of the polarization vectors of the scalar, respectively the vector field and \( \mathbf{q} \) and \( \varepsilon \) are the frequency and the wave vector of these modes. Substituting this Ansatz into the Lorentz condition, Eq. 13 we find \( \varepsilon e_\phi + \mathbf{q} \cdot e_\mathbf{A} = 0 \). This leaves us with three independent modes out of the four we have started with. But the Lorentz condition itself, having to be invariant under the electromagnetic gauge transformations, permits us to reduce the number of independent modes even further. We can choose for that any one of the components of \([\phi(x, t), A(x, t)]\) to be eliminated. Let us choose it such that the time component \( \phi(x, t) \) vanishes. This is achieved by transforming Eq. 13 by the gauge transformation \( \phi(x, t) \rightarrow \phi'(x, t) = \phi(x, t) - \frac{1}{c} \frac{\partial}{\partial t} \Lambda(x, t) = 0 \), with \( \Lambda(x, t) = i(\varepsilon e_\phi / \varepsilon) e^{-i(q \cdot x - ct)} \). Together with the constraint \( \varepsilon e_\phi + \mathbf{q} \cdot e_\mathbf{A} = 0 \), we are thus left with \( \mathbf{q} \cdot e_\mathbf{A} = 0 \), which leaves us with two independent modes, orthogonally polarized to the direction of the wave propagation \( \mathbf{q} \).

This of course only reiterates the general quantum field theoretical argument that the excitations of a spin-1 vector gauge boson field, such as the electromagnetic field, can neither have finite mass nor a longitudinal component. The
electromagnetic force fields, thus having infinite range, implies that transverse modes of the electromagnetic field propagate freely in vacuum. The magnetic field, transported by these modes therefore freely transverses a metal in its non-superconducting state, apart from a minimal Landau diamagnetic response to it.

This is no longer the case when this metal becomes a superconductor. Bringing the electromagnetic field into contact with a supercurrent, the transverse photons can no longer propagate freely, as evidenced by the Meissner-Ochsenfeld magnetic field screening. As we have seen in the previous section the Maxwell equations then have to be replaced by the Proca equations, Eqs. [13] Rephrasing these equations in terms of the vector and scalar fields, in analogy to the Maxwell equations in vacuum, Eqs. [14] and putting the source terms $\propto \rho, \mathbf{J}$ equal to zero, we obtain the quantum field theory equations for spin-1 vector gauge fields:

$$\left( \nabla^2 + \frac{m_0^2 c^2}{\hbar^2} \right) \phi(x, t) - \frac{1}{c^2} \partial_t^2 \phi(x, t) + \frac{1}{c} \partial_t \left( \nabla \mathbf{A}(x, t) + \frac{1}{c} \partial_t \phi(x, t) \right) = 0$$

$$\left( \nabla^2 + \frac{m_0^2 c^2}{\hbar^2} \right) \mathbf{A}(x, t) - \frac{1}{c^2} \partial_t^2 \mathbf{A}(x, t) - \nabla \left( \nabla \phi(x, t) + \frac{1}{c} \partial_t \phi(x, t) \right) = 0. \quad (17)$$

The introduction of the mass term in these equations qualitatively changes the spectral properties of the excitations of these gauge fields. In order to see that, let us investigate what happens to the polarization vectors of $\phi(x, t)$ and $\mathbf{A}(x, t)$ when passing from the massless spin-1 vector boson fields to massive ones. Operating onto the first of these two equations, Eqs. [17] with $-\frac{1}{c} \partial_t$ and on the second equation with $\nabla$ and subsequently summing the two, we have

$$\frac{m_0^2 c^2}{\hbar^2} \left( \frac{1}{c} \partial_t \phi(x, t) + \nabla \mathbf{A}(x, t) \right) = 0. \quad (18)$$

This implies that the Lorentz condition is automatically satisfied and therefore can not serve to reduce the number of independent modes from four to three, as was the case of the Maxwell equations, Eqs. [14] describing massless photons. Implementing Eq. [18] Eqs. [17] are in fact intrinsically equivalent, i.e.,

$$\left( \nabla^2 - \frac{1}{c^2} \partial_t^2 + \frac{m_0^2 c^2}{\hbar^2} \right) \phi(x, t) = 0$$

$$\left( \nabla^2 - \frac{1}{c^2} \partial_t^2 + \frac{m_0^2 c^2}{\hbar^2} \right) \mathbf{A}(x, t) = 0. \quad (19)$$

But using, as before, the Ansatz $\phi(x, t) = e^\phi e^{-i(q \cdot x - ct)}$ and $\mathbf{A}(x, t) = e^\mathbf{A} e^{-i(q \cdot x - ct)}$, we obtain from Eq. [18] $[\varepsilon \varepsilon_\phi - \mathbf{q} \cdot \mathbf{e}_A = 0]$, which reduces the number of modes from four to three independent modes. But unlike in the case of the Maxwell equations in vacuum, Eqs. [14] which are invariant under gauge transformations, Eqs. [19] are not any longer. Hence no gauge transformation can reduce the number of independent modes any further. We thus have two massive transverse circularly polarized modes with polarization vectors $\mathbf{e}_\perp = \frac{1}{\sqrt{2}} [0, 1, \pm i, 0]$ and one longitudinal massive mode with a polarization vector $\mathbf{e}_\parallel = (c^2 q_0^2 + \varepsilon^2)^{-1/2} [c q_z, 0, 0, \varepsilon]$, which can be considered as a helicity-0 component spin-1 vector gauge boson.

The equations [13] which result from combining the Maxwell equations with the London equations, represent the deep physical content of the Meissner-Ochsenfeld effect. Transposing these intricate electromagnetic features of the spontaneously broken symmetry of the phase locked state of a superconductor, onto that of the matter field, described by a polarized quantum vacuum in contact with a massless spin-1 boson vector field, culminated in the emergence of massive gauge fields in the standard model of elementary particle physics via the Anderson-Higgs mechanism.

Before illustrating in the next two sections how mass production in the force-transmitting excitations of the electromagnetic field comes about, let us look at a simpler situation, namely that described by the relativistic wave equation for spin-0 particles, i.e., the Klein-Gordon wave equation

$$\left( \nabla^2 - \frac{1}{c^2} \partial_t^2 + \frac{m_0^2 c^2}{\hbar^2} \right) \Phi(x, t) = 0. \quad (20)$$

This equation was proposed around 1926 simultaneously by different authors as a relativistic generalization of the Schrödinger equation for a single particle. The mass production here can be envisaged as arising from a breaking of a continuous translational symmetry, such as that experienced in a chain of atoms coupled with each other on nearest neighbor sites. This is a perfectly continuous translational invariant system, whose modes are the acoustic phonons with an energy given by $\omega_q = v q$. But when such a chain of atoms is embedded in a substrate, in such a way that the displacement of the atoms are coupled to fixed position in a substrate, the translational symmetry is broken and the
excitations will have acquired a finite mass. The Hamiltonian for such a scenario is given by

\[ H_{\text{Lattice}} = \sum_{<i,j>} \frac{1}{2} \left[ m \dot{u}_i^2 + \omega^2 (u_i - u_j)^2 + \omega_0^2 u_i^2 \right]. \]  

In the continuous limit it becomes

\[ H_{\text{Lattice}} = \int dx \frac{1}{2} \left[ m (\dot{\Phi}(x,t))^2 + v^2 (\nabla \phi)^2 (x,t) + \omega_0^2 \Phi^2 (x,t) \right]. \]

\[ v = \omega a \text{ and } \Phi(x,t) = u_i(t) \sqrt{a}. \]  
The interatomic distance \( a \) of the lattice has to be taken equal to zero as we approach the continuum limit. The excitations of this system are then described by the dynamical lattice deformations, having the energies \( \omega \Phi = \sqrt{\omega^2 q^2 + \omega_0^2} \) with \( \omega_0^2 \) playing the role of the mass which is induced by the polarization of the acoustic phonons due to their coupling to the broken symmetry of the substrate. As we shall see in the next two sections, the mass of the photons accompanying the super-current feel the rotational symmetry breaking of the superconducting phase, which, like in the example given here, couples to the translational massless modes of the electromagnetic field and thereby renders them massive.

V. THE EMERGENCE OF MASSIVE GAUGE BOSONS - THE ANDERSON-HIGGS MECHANISM AND ITS FIELD THEORETICAL TREATMENT

By the mid fifties, the Lorentz covariant gauge-field theoretical approach in elementary particle physics was well on the way. It was based on the concept of a quantum vacuum, composed of a multitude of degrees of freedom, which, via the fluctuations of symmetry broken polarized states, could give these degrees of freedom a true physical significance. The resulting excitations represent the fermionic elementary particles: six quarks and the six leptons: the electron, the muon and the tau, together with their corresponding neutrinos. A particular role in this approach is played by the dynamical breaking of continuous symmetries, via the so called spontaneous symmetry breaking (SSB). It results in the emergence of bosonic collective Nambu-Goldstone modes. The associated to them excitations are manifest in form of conserved currents, which reflect the basic conservation laws in particle physics to which we have alluded to in the Introduction and which derive from ”internal symmetries” of the quantum vacuum. The Goldstone theorem stipulates that the Nambu-Goldstone modes should be inherently massless, inspired by the concept of Cooper pairs, made out of itinerant electrons in the BCS theory, led Nambu and Jona-Lasinio to propose Dirac fermions as composites of mesons and nucleons as a result of chiral symmetry breaking.

The introduction of quantized force-transmitting gauge fields (the photon for the electromagnetic forces, the gluons for the strong nuclear forces and the \( W^\pm \) and \( Z^0 \) for the electro-weak forces), which accompany the elementary particles, assure that the conservation laws are respected in scattering and decay processes. By construction, these gauge fields are massless bosons. But with the exception of the massless photons, the remaining gauge bosons all are massive. In the example of the \( \beta \) decay, the mass of the force-transmitting \( W^\pm \) and \( Z_0 \) gauge bosons amount to roughly 80 GeV. The intrinsic masslessness of the force-transmitting gauge boson, let alone that of the fermionic elementary particles in such an approach, presented a considerable stumbling block for these gauge field theories.

A way out of this dilemma was proposed by Julian Schwinger. He suggested that local current conservation of a conserved quantity could still be maintained provided one associates to it a gauge transformation, which simultaneously acts onto the gauge fields, accompanying such currents. It is in this way that gauge fields acquire mass.

Phil Anderson came up with a physical realization for such a scenario in terms of a non-relativistic analog: a plasma of a free electron gas, in which gauge invariance as well as particle conservation are assured, even though the associated to it gauge photons are not collective low frequency modes. The propagating transverse and longitudinal plasma modes, with frequencies above the plasma frequency, play in this scenario the role of Schwinger’s conjectured massive vector gauge bosons. The density fluctuations of such an electron plasma display an infinite polarizibility which is tantamount to the polarizibility of the matter field, i.e., the quantum vacuum fluctuations in particle physics. The longitudinal plasma mode, corresponding to the helicity zero gauge boson in Lorentz gauge invariant field theory, would be absent if one could switch off its gauge coupling to the density fluctuations of the electron gas, respectively the symmetry broken vacuum fluctuations in quantum field theory. But since the longitudinal plasma modes couples via gauge transformations to a background electromagnetic gauge field, it thereby switches on the otherwise forbidden longitudinal component of this gauge field. It is this, which subsequently induces a mass in the transverse modes and renders the passage of transverse electromagnetic waves in such a medium opaque, when their frequency is below that of the plasma mode. This scenario for introducing massive excitations into the Yang-Mills gauge theories was subsequently transposed onto Lorentz covariant gauge field theories by Higgs and independently by Englert and Brout as well as Guralnik, Hagen and Kibble, which significantly helped to consolidate the standard model.
The Anderson-Higgs mechanism is based on a self-consistent dielectric mechanism in metals in which current fluctuations $J(q, \varepsilon)$, monitored by the plasma oscillations, act as source terms in the associated electromagnetic field $A^m(q, \varepsilon)$

$$A^m(q, \varepsilon) = \frac{1}{[\varepsilon^2 q^2 - q^2]} J^m(q, \varepsilon).$$

But since the current fluctuations themselves are monitored by the linear response $K^{mn}(q, \varepsilon)$ to this electromagnetic field $A^m(q, \varepsilon)$, it follows from the classical dielectric response in metals\textsuperscript{65} that

$$J^m(q, \varepsilon) = -K^{mn}(q, \varepsilon) A^n(q, \varepsilon),$$

where $\varepsilon$ and $q$ denote the frequency and wavevectors of these fields. In the static limit ($\varepsilon \to 0$), gauge invariance imposes the structure of the kernel: $K^{mn}(q,0) = (q^m q^n - \delta^{mn} q^2) K(q^2)$. With the choice of the London gauge, i.e., $q \cdot A(q, \varepsilon) = 0$, this reduces to $K^{mn}(q,0) = -\delta^{mn} n e^2 / m e_c$. $n_e$ and $m_e$ denote the density and mass of the charge of the electrons. Substituting this expression for $K(q^2)$ into the above two coupled equations, results in the wave equations for the three components of the electromagnetic vector field

$$\left(c^2 q^2 - \varepsilon^2 + \frac{n e^2}{m e_c}\right) A^m(q, \varepsilon) = 0,$$

when in contact with the electrons which make up the plasma.

The Anderson-Higgs mechanism, in the example of an electron plasma, hinges on two conditions: (i) The fluctuating electron plasma (the matter-field, mimicking the quantum vacuum with a corresponding Nambu-Goldstone mode in quantum field theory), must have an intrinsic infinite polarizibility before being coupled to any force-transmitting gauge field. (ii) The physical manifestations of the in principle unobservable underlying symmetry of the matter field are apparent in the conserved currents, which correlate the dynamics of the matter-field and the gauge fields,\textsuperscript{30} in an overall electromagnetic gauge invariant manner.

Let us conclude this section by the field theoretical formulation of the Anderson-Higgs mechanism for converting massless gauge fields into massive ones. We follow for that purpose the well presented discussion of it, given in ref.\textsuperscript{68}. Our illustration here is based on a paradigm for superconductivity in terms of a complex scalar field, which plays the role of an order parameter of a SSB state and which is locally coupled to an electromagnetic field. This paradigm represents the Ginzburg-Landau phenomenology of a superconductor, described by a macroscopic condensate wave function

$$\Psi^*(x) = e^{-i\varepsilon(x)}|\Psi(x)| \quad \Psi(x) = e^{i\varepsilon(x)}|\Psi(x)|.$$  

(26)

We assume the SSB superconducting state to be parametrized by a Mexican hat potential

$$V[\Psi^*(x), \Psi(x)] = -|\alpha|^2 \Psi^*(x) \Psi(x) + |\beta|^2 \Psi^*(x) \Psi(x)^2.$$  

(27)

and that the condensate wave function is coupled to a locally gauge invariant four component electromagnetic gauge field $A^\mu(x)$. The density fluctuations of such a superfluid condensate $\delta \Psi(x)$ around a fixed average amplitude $|\Psi(x)| = \Psi_0 = \sqrt{n_s} = \sqrt{\alpha/\beta}$ and the phase fluctuations $\delta \Theta(x)$ around a symmetry broken state, given by an arbitrary but fixed phase, $\Theta_0$, are parametrized by

$$\Psi(x) = e^{i(\Theta_0 + \delta \Theta(x))}(\Psi^0 + \delta \Psi(x))$$

and

$$\Psi^*(x) = e^{-i(\Theta_0 + \delta \Theta(x))}(\Psi^0 + \delta \Psi(x)).$$

(28)

The Lagrangian for the superconductor matter-field, coupled to the electromagnetic gauge field in a relativistically covariant Minkowski metric $[x^\mu = \{ct, x, y, z\}, \ x_\mu = \{ct, -x, -y, -z\}$, is (in natural units $\hbar = c = 1$) given by

$$L[\Psi^*(x), \Psi(x), A(x)] = \left[(\partial^\mu + i e A^\mu(x)) \Psi(x)^* \right] \left[(\partial_\mu + i e A_\mu(x)) \Psi(x) \right] + V[\Psi^*(x), \Psi(x)] - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}.$$  

(29)

$L[A(x)] = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ denotes the Lagrangian for the electromagnetic field in terms of the electromagnetic field tensor $F^{\mu\nu} = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$, where $A^\mu(x)$ satisfies the Maxwell equations:

$$\frac{\delta L[A(x)]}{\delta A^\mu(x)} = 0 \rightarrow \partial^\nu F_{\nu\mu} = 0.$$

(30)
and whose solutions are given by two orthogonally polarized transverse modes. The longitudinal mode is prohibited by gauge invariance of these equations under the transformations $A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$ and imposing the Lorentz condition $\partial_\mu A^\mu(x) = 0$, as we have illustrated in the previous section IV.

In the absence of any electromagnetic gauge field (setting $A(x) = F^{\mu\nu}(x) = 0$ in the above Lagrangian), we obtain the dynamics of the phase and amplitude fluctuations by developing this Lagrangian to second order in $\delta \Psi(x)$ and $\delta \Theta(x)$ around the stationary solution $|\Psi(x)|^2 = \Psi_0^2 = n_s = -\alpha/\beta$ and $\Theta(x) = \Theta_0$ denoting an arbitrary but fixed phase between 0 and $2\pi$:

$$\partial^\mu \partial_\mu \delta \Theta(x) = 0 \quad (\partial^\mu \partial_\mu - 2\alpha) \delta \Psi(x) = 0 \Rightarrow (-E^2 + p^2 + m_0^2) \delta \Psi(x) = 0.$$  
(31)

(32)

Its solutions are given by a massive amplitude mode $\delta \Psi(x)$ with a mass $m_0^2 = 2|\alpha| = 2n_s\beta$ and a massless Nambu-Goldstone mode $\delta \Theta(x)$.

This massless Nambu-Goldstone mode is eliminated, when the spontaneously broken matter field is coupled to the electromagnetic field through the first term in Lagrangian, Eq. (29). A simple gauge transformation

$$\Psi(x) \Rightarrow \tilde{\Psi}(x) = e^{-i\Phi(x)} \Psi(x) \Rightarrow \Psi_0 + \delta \Psi(x)$$

$$\Psi^*(x) \Rightarrow \tilde{\Psi}^*(x) = e^{+i\Phi(x)} \Psi^*(x) \Rightarrow \Psi_0 + \delta \Psi(x)$$

$$A^\mu(x) \Rightarrow \tilde{A}^\mu(x) = A^\mu(x) + (1/e)\partial^\mu \Phi(x) \Rightarrow A^\mu(x) + (1/e)\partial^\mu \Theta(x)$$

(33)

can do that, given the fact that the electromagnetic field tensor is locally gauge invariant. The original Lagrangian, Eq. (29) then is transformed into

$$L[\tilde{\Psi}^*(x), \tilde{\Psi}(x), \tilde{A}(x)] = \left[ \partial^\mu - ie\tilde{A}^\mu(x) \right] \left( \Psi_0 + \delta \Psi(x) \right) \left[ \partial_\mu + ie\tilde{A}_\mu(x) \right] \left( \Psi_0 + \delta \Psi(x) \right)$$

$$\left[ -\alpha \left( \Psi_0 + \delta \Psi(x) \right)^2 - \frac{\beta}{2} \left( \Psi_0 + \delta \Psi(x) \right)^4 - \frac{1}{4} F^{\mu\nu} \tilde{F}_{\mu\nu} \right].$$

(34)

(35)

The massless phase fluctuations of the Nambu-Goldstone mode, characterizing the spontaneously broken symmetry of the superconductor matter-field before its coupling to the electromagnetic fields, have thus dropped out in this procedure. They have been absorbed - "eaten up" - by the excitations of a renormalized electromagnetic field $\tilde{A}_\mu(x)$. Variation of $L[\tilde{\Psi}^*(x), \tilde{\Psi}(x), \tilde{A}(x)]$ with respect to $\{\delta \Psi(x), \tilde{A}(x)\}$ leads to

$$\frac{m_0^2 e^2}{\beta} \tilde{A}^\mu(x) + \partial_\nu F^{\nu\mu}(x) \Rightarrow \left( \partial_\nu \partial^\nu - \frac{m_0^2 e^2}{\beta} \right) \tilde{A}^\mu(x) = 0,$$

(36)

(37)

with the second equation representing the Proca equation for massive electromagnetic fields. It describes the emergence of the massive longitudinal mode and the transformation of the massless transverse photons into massive ones. For obtaining this equation, we have exploited the identity

$$\partial_\nu F^{\nu\mu}(x) = \partial_\nu \partial^\nu A^\mu(x) - \partial^\mu \partial_\nu A^\nu(x) \equiv \partial_\nu \partial^\nu A^\mu(x)$$

(38)

which derives from the Lorentz condition $\partial_\nu \tilde{A}^\mu(x) = 0$. This reduces the number of independent components of $\tilde{A}^\mu(x)$ to three. No further gauge transformation can reduce the number of solutions to below that, as follows from general field theoretical arguments for massive spin-1 vector gauge bosons, satisfying the above Proca equation for electrodynamics that have lost their gauge invariance because of the mass term.

**VI. DERIVATION OF THE LONDON EQUATIONS FOR A CHARGED CONDENSED BOSE LIQUID AND OF ITS ELECTRO-MAGNETIC PROPERTIES**

In the previous two sections we have addressed the emergence of the super-current and its associated to it massive electromagnetic field from two diametrically opposite points of view:

(i) From the classical phenomenological point of view, based on the London equations, in which the steady state of the diamagnetic super-current arises from a qualitative modification of the electromagnetic field, which is irrevocably locally coupled to this current. Its frictionless transport can only occur in conjunction with Meissner-Ochsenfeld screening, which not only results from this current but sustains it.
(ii) From a quantum field theory point of view, in which a symmetry broken quantum vacuum constitutes a matter field, whose collective excitations (the Nambu-Goldstone modes) are coupled to the quantized excitations of the electromagnetic field in which the superconductor matter field is embedded. The correlated dynamics of the charge current of the matter field and the electromagnetic spin-1 vector boson gauge field transforms, via the Anderson-Higgs mechanism, the spin-0 scalar Nambu-Goldstone mode into a massive electromagnetic helicity-0 (longitudinal) spin-1 vector gauge boson.

In the classical description, in terms of the London equations, the concept of a phase of the superfluid is totally absent. In spite of that, as we have seen in Sec. IV, this approach ultimately leads to transform the Maxwell equations into Proca equations for massive electromagnetic field excitations which locally accompany the supercurrent and thereby break electromagnetic gauge invariance. In the field theoretical description one assume from the outset a fixed phase of the macroscopic Ginzburg-Landau wave function for the superconducting ground state before the electromagnetic field is switched on. The necessity of such an assumption and thereby of a SSB scenario for a solid state system has been questioned since superconductivity is characterized by an out of equilibrium state flow of the supercurrent, which ultimately relies on the sole fact that the electromagnetic field accompanying it has become massive. Phil Anderson and Anthony Leggett together with Fernando Sols have addressed the question to what extent the phase-locking of a superfluid condensate is a necessary or even a "meaning full concept" for an isolated superconductor.

A similarly ambiguous situation was encountered when, with Walter Thirring in 1962, we derived the London equations within a linear response function approach for a paradigm of superconductivity: the charged Bose liquid. At that memorable time, the forthcoming of fundamentally new concepts in physics was at its height. Although it addressed primarily the field theoretical aspects of superconductivity, foremost by Goldstone, Weinberg and Salam, by Nambu and Jona-Lasinio, by Schwinger, it was widely shared in the physics community at large, which was at ease to navigate between solid state, nuclear and particle physics. Simultaneously, crucial experimental results a that time pinned down the physical significance of the phase of the superconducting condensate in terms of flux quantization, and the Josephson tunneling.

What we found in our study with Walter Thirring was that independent on whether these bosons are interacting or not, one arrives at the same London equations. In the first case, the linear in momentum Bogoliubov modes of the interacting bosons suggest the existence of Nambu-Goldstone modes and hence a SSB of a continuous phase variable or not, one arrives at the same London equations. In the first case, the linear in momentum Bogoliubov modes of the interacting bosons suggest the existence of Nambu-Goldstone modes and hence a SSB of a continuous phase variable of the condensate. In the second case, their free particle like spectrum simply makes the bosons condense into a Bose-Einstein condensate for which the phase of the particles is not fixed. It merely requires that the single-particle density matrix has a single macroscopic eigenvalue.

Let us now sketch the physics which we unearthed in this work and extend it in order to bring it into line with the field theoretical approach to superconductivity, discussed in the previous section.

The Hamiltonian for an interacting Bose liquid, exposed to an electromagnetic vector field \( \mathbf{A}(x,t) \) is given by

\[
H = H_0 + H_A, \quad \text{with}
\]

\[
H_0 = \int d^3 x \psi^\dagger(x,t) \left( -\frac{\hbar^2 \nabla^2}{2M} \right) \psi(x,t) - \frac{1}{2} \psi(x,t) \psi(x,t) \nabla^\mu \nabla_\mu \psi(x,t) \psi(x,t) \]

\[
H_A = \frac{e^2}{2M c^2} \int d^3 x \left[ \psi^\dagger(x,t) \left( \frac{\hbar}{i} \right) \nabla \psi(x,t) - \psi(x,t) \left( \frac{\hbar}{i} \right) \nabla \psi^\dagger(x,t) \right] \mathbf{A}(x,t) + \frac{e^2}{2M c^2} \mathbf{A}^2 \psi^\dagger \psi.
\]  

In a typical setting of a Meissner-Ochsenfeld experiment, the circulating current in a superconducting ring or a hollow metallic sphere, is triggered by a magnetic field. It implies the action of exclusively the transverse component of the electromagnetic vector field \( \mathbf{A}(x,t) \), given by the magnetic field \( \mathbf{B}(x,t) = \nabla \times \mathbf{A}(x,t) \). Localized charges, which potentially could act as sources for an electric field \( \mathbf{E}(x,t) = -\frac{1}{\epsilon} \partial_t \mathbf{A}(x,t) + \nabla \phi(x,t) \) are absent in such a setup. The scalar potential \( \phi(x,t) \) of the electromagnetic field thus can be put equal to zero, together with the constraint for the electromagnetic gauge: \( \partial_x \mathbf{A}(x,t) = 0 \).

The current of the bosonic charge carriers is composed of a magnetic field-induced kinetic contribution \( \mathbf{J}_{kin}(x,t) \) and a diamagnetic contribution, deriving from the associated to it induced charge density fluctuations \( \epsilon \rho(x,t) = \psi^\dagger(x,t) \psi(x,t) \), as given by

\[
\mathbf{J}(x,t) = \mathbf{J}_{kin}(x,t) - \frac{e^2}{M c} \mathbf{A}(x,t) \rho(x,t),
\]

\[
\mathbf{J}_{kin}(x,t) = -\frac{e^2}{2M c} \left[ \psi^\dagger(x,t) \left( \frac{\hbar}{i} \right) \nabla \psi(x,t) - \psi(x,t) \left( \frac{\hbar}{i} \right) \nabla \psi^\dagger(x,t) \right].
\]

That latter is correlated to the fluctuations of the induced current via the continuity equation and controls charge current conservation.
The driving mechanism for achieving a self-sustained conserved "super-current" is contained in the linear response of the Bose condensed system to an associated to it local source field $A(x, t)$

$$
\langle J^m(x, t) \rangle_A = -\frac{i}{\hbar} \int_{-\infty}^{t} d^3x' dt' \langle [J^m_{kin}(x, t), H_A(x', t')] \rangle - \frac{e^2}{Mc} A^m(x, t) \langle \rho(x, t) \rangle 
$$

$$
= \int_{-\infty}^{t} d^3x' dt' K^{mn}(x - x', t - t') A^n(x', t') 
$$

$$
K^{mn}(x - x', t - t') = \frac{i}{\hbar c} \Theta(t - t') \langle [J^m_{kin}(x, t), J^n_{kin}(x', t')] \rangle - \frac{e^2}{Mc} \delta^{mn} \delta(x - x') \delta(t - t') \langle \rho(x, t) \rangle, 
$$

similar to Phil Anderson’s treatment of the electron plasma discussed in the previous section V. In order for the source field $A(x, t)$ to be sustained by the flow of the conserved total current $J(x, t)$ it has to satisfy the Maxwell wave equations with a source term $\langle J(x, t) \rangle_A$, satisfying the equation

$$
\left[ \nabla^2 - \frac{1}{\epsilon^2} \partial_t^2 \right] A(x, t) = -\langle J(x, t) \rangle_A. 
$$

Charge conservation in this feed back process between the induced kinetic current $\langle J(x, t) \rangle_A$ and the intrinsically diamagnetic one, $(e^2/Mc)A(x, t)$, is assured by relating them via the continuity equation. This was evidently missing in Felix Bloch’s initial proposition, Eq. 9 in Sec. IV, but was introduced in terms of a contribution $J_{coll} = \nabla f(x, t)$ coming from collective modes in the superconducting state by Nambu as we have illustrated.

In our treatment of the London equations in ref. 34 we assured a manifestly gauge invariant description of the charged Bose fluid, embedded in an externally applied electromagnetic field by imposing the longitudinal sum rule, which derives from the canonical commutation relations

$$
\langle [J^m_{kin}(x, t), \rho(x', t')] \rangle = -\frac{e}{2M} \left( \frac{\hbar}{i} \right) \frac{d}{dx_m} \delta(x - x') \delta(t - t') \langle 2\rho(x, t) \rangle, 
$$

$$
\frac{d}{dx_m} \frac{d}{dx_n} \langle [J^m_{kin}(x, t), J^n_{kin}(x', t')] \rangle = -\frac{e^2}{2M} \left( \frac{\hbar}{i} \right) \nabla^2 \delta(x - x') \partial_t \delta(t - t') \langle 2\rho(x, t) \rangle. 
$$

In order to render transparent the mutual inter-relation between the conserving current of an incipient superconductor (in the absence on any electromagnetic field) and the associated to it electromagnetic field, when it interacts with the charge carriers, we separate the kernel $K^{mn}(x, t)$ in Eq. 44 into its longitudinal $(\propto A(x, t))$ and transverse $(\propto \nabla \times \nabla \times A(x, t))$ response:

$$
K^{mn}(q, \epsilon) = \delta^{mn} \frac{e^2}{2Mc} \int d\epsilon' [f_{\|}(q, \epsilon') - f_{\|}(q, -\epsilon')] \left[ \frac{1}{\epsilon' - \epsilon - i\eta} - \frac{1}{\epsilon' + i\eta} \right] 
$$

$$
+ (q^m q^n - q^2 \delta^{mn}) \frac{e^2}{2Mc} \int d\epsilon' [f_{\perp}(q, \epsilon') - f_{\perp}(q, -\epsilon')] \left[ \frac{1}{\epsilon' - \epsilon - i\eta} \right]. 
$$

Expressing the boson wave functions in terms of boson creation and annihilation operators $\psi(x, t) = \Omega^{-\frac{i}{2}} \sum_k b_k^e e^{-ik\cdot x}$, $\psi^*(x, t) = \Omega^{-\frac{i}{2}} \sum_k b_k^e e^{ik\cdot x}$ (with $\Omega$ denoting the normalization volume), we have

$$
f_{\|}(q, \epsilon) = \frac{4}{d^2} \sum_{k, k', z, z'} \left\{ \frac{1}{2} e^{\frac{i(c_u + \mu N)}{q^z}} (q \cdot k)(q \cdot k') \right\} 
$$

$$
\times \langle z| b_{-k+\frac{i}{2}q}^e b_{k+\frac{i}{2}q}^e |z'\rangle \langle z'| b_{-k+\frac{i}{2}q}^e b_{k+\frac{i}{2}q}^e |z\rangle \delta(\epsilon - E_z + E_z') 
$$

$$
f_{\perp}(q, \epsilon) = \frac{2}{d^2} \sum_{k, k', z, z'} \left\{ \frac{1}{2} e^{\frac{i(c_u + \mu N)}{q^z}} \left( \frac{2(q \cdot k)(q \cdot k')}{q^2} - (k \cdot k') \right) \right\} 
$$

$$
\times \langle z| b_{-k+\frac{i}{2}q}^e b_{k+\frac{i}{2}q}^e |z'\rangle \langle z'| b_{-k+\frac{i}{2}q}^e b_{k+\frac{i}{2}q}^e |z\rangle \delta(\epsilon - E_z + E_z'). 
$$

The second term in the brackets for the longitudinal component of the kernel in Eq. 48 derives from the purely diamagnetic contribution to the current. Exploiting the longitudinal sumrule, Eq. 47 we cast it into the form

$$
2 \rho(x, t) = \frac{e}{2M} \int \frac{d\epsilon'}{\epsilon'}\left( f_{\|}(q, \epsilon') - f_{\|}(q, -\epsilon') \right). 
$$
The first term in Eq. (48) describes the response of the induced current to its generating electric source field \( \mathbf{E}(x, t) \) and the second term to its generating magnetic source field \( \mathbf{B}(x, t) \). This formulation satisfies the longitudinal and transverse sum rules.\(^2\) Eq. (48) is a generalization of an earlier formulation by Schafroth,\(^2\) who focused on the effect of Meissner-Ochsenfeld screening in the static limit and therefore restricted himself to the London gauge, i.e., \( \nabla \mathbf{A}(x, t) = 0 \). Within such a choice of electromagnetic gauge fixing, the longitudinal response, which introduces the longitudinal massive photons, can not be obtained.

The matrix elements in the kernel Eq. (48) for the Bose liquid, described by the Hamiltonian Eqs. (39) and (40) have been evaluate in our paper, ref. 46\(^1\) for the free Bose liquid, the weakly interacting one and for electrons described by the BCS scenario, after having imposed gauge currents to render the BCS Hamiltonian gauge invariant. For the non-interacting free Bose liquid \( (g = 0 \text{ in the Hamiltonian, Eq. (39)}, \) the space-time Fourier transform of the kernel becomes

\[
K^{mn}(\mathbf{q}, \varepsilon) = \frac{e^2}{2Mc} \sum_k \frac{N_k - \frac{1}{2} \mathbf{q}}{E_k + \frac{1}{2} \mathbf{q}} - \frac{N_k - \frac{1}{2} \mathbf{q}}{E_k - \frac{1}{2} \mathbf{q}} - \varepsilon \left[ \frac{\delta^{mn} 2\varepsilon (\mathbf{k} \cdot \mathbf{q})}{q^2} + \left( \frac{\delta^{mn} - q^m q^n}{q^2} \right) 2 \left( k^2 - \frac{3(k \cdot q)^2}{q^2} \right) \right]
\]

with \( E_k \pm \frac{1}{2} \mathbf{q} = \frac{k^2}{2M}(k \pm \frac{1}{2} \mathbf{q})^2 \) and \( N_k \pm \frac{1}{2} \mathbf{q} = (\exp(E_k \pm \frac{1}{2} \mathbf{q}) - \mu)/k_BT \) denoting the Bose distribution function for a boson density controlled by the chemical potential \( \mu \). Given the feature of Bose-Einstein condensation with a density of zero momentum bosons \( N_0 \), we substitute \( N_k \pm \frac{1}{2} \mathbf{q} \) in Eq. (52) by \( N_0(T) \delta_k \pm \frac{1}{2} \mathbf{q} \). The second term represents the non-condensed fraction of the bosons, which for a free non-interacting Bose liquid tends to zero as we approach zero temperature. The expectation value of the current, Eq. (43) then becomes

\[
\langle J^m(q, \varepsilon) \rangle_A = \frac{N_0 e^2}{Mc} \left[ \frac{\delta^{mn} \varepsilon^2}{\omega_q^2 - \varepsilon^2} A^m(q, \varepsilon) - \frac{\omega^2}{q^2} \frac{1}{\omega_q^2 - \varepsilon^2} c \right] \left( \frac{\delta^{mn} - q^m q^n}{q^2} \right) A^n(q, \varepsilon),
\]

where \( \omega_q = (\hbar^2 q^2/2M) \). With \([\text{rot rot} \mathbf{A}](q, \varepsilon) = [\text{rot} \mathbf{B}](q, \varepsilon) \) and \( i\varepsilon \mathbf{A}(q, \varepsilon) = c \mathbf{E}(q, \varepsilon) \), we find

\[
\langle J(q, \varepsilon) \rangle_A = \frac{N_0 e^2}{2M} \frac{i \varepsilon}{\omega_q^2 - \varepsilon^2} \mathbf{E}(q, \varepsilon) - \frac{N_0 e^2}{2Mc} \frac{\omega^2}{\omega_q^2 - \varepsilon^2} \left( \frac{1}{q^2} \right) \left[ \text{[rot} \mathbf{B}\text{]}(q, \varepsilon)
\]

\[
= \sigma(q, \varepsilon) \mathbf{E}(q, \varepsilon) + \chi(q, \varepsilon) [\text{rot} \mathbf{B}](q, \varepsilon),
\]

with

\[
\sigma(q, \varepsilon) = \frac{N_0 e^2}{2M} \frac{i \varepsilon}{\omega_q^2 - \varepsilon^2}, \quad \chi(q, \varepsilon) = -\frac{N_0 e^2}{Mc} \left[ \frac{1}{q^2} \right] \frac{\omega^2}{\omega_q^2 - \varepsilon^2}
\]

denoting the electric conductivity and magnetic susceptibility respectively. Their limiting behavior

\[
\sigma(q = 0, \varepsilon \to 0) = D \delta(\varepsilon), \quad D / e^2 = \frac{N_0}{M} \pi
\]

\[
\chi(q \to 0, \varepsilon = 0) = -Ds / q^2, \quad D_s / e^2 = \frac{N_0}{Mc}
\]

provides us with (i) the Drude weight \( D \), given in terms of the density to mass ratio of itinerant charge carriers and (ii) the superfluid weight \( D_s \) in terms of the density to mass ratio of the superfluid charge carriers. For the non-interacting free Bose liquid we obtain at \( T = 0 \) equal weights \( N_0/M \) for \( D \) and \( D_s \), which is the hallmark of its superfluid ground state.\(^2\) As one approaches the superfluid transition temperature, \( N_0 \) tends to zero. \( N_0 \) is then given exclusively by the distribution function of the non-condensed bosons \( N_k = (\exp(E_k/k_BT) - 1)^{-1} \). In the long wavelength limit, the weight of the transverse component of the kernel, controlling the singular behavior of the static \( (\varepsilon = 0) \) magnetic susceptibility then tends to zero in the angle averaged kernel as

\[
\lim_{q \to 0} \sum_k \frac{N_k - \frac{1}{2} \mathbf{q}}{E_k + \frac{1}{2} \mathbf{q}} - \frac{N_k - \frac{1}{2} \mathbf{q}}{E_k - \frac{1}{2} \mathbf{q}} - \varepsilon \left( \frac{k^2 q^2 - 3(k \cdot q)^2}{q^2} \right) = 0.
\]

The contribution to the kernel arising from the longitudinal part, Eq. (49) on the contrary, remains finite with a Drude weight, which characterizes a Bose metal state above \( T_c \).
The expression for the super-current $\langle J(q, \varepsilon) \rangle_A$ in Eq. (53) contains all the ingredients necessary for deriving the London equations. Multiplying it by $i\varepsilon = \partial / \partial t$ (giving us its time derivative), respectively applying the operator $-i\mathbf{q} \times \text{rot}$ (giving us its rotational component), we obtain

$$i\varepsilon \langle J^m(q, \varepsilon) \rangle_A = \frac{N_0 e^2}{M} \left( \delta_{mn} - \frac{\omega_q^2}{\omega_q^2 - \varepsilon^2} q_m q_n \right) i\varepsilon E^m(q, \varepsilon) = \frac{N_0 e^2}{M} E^m(q, \varepsilon)$$

(59)

$$- i[\mathbf{q} \times \langle J(q, \varepsilon) \rangle_A]^m = \frac{N_0 e^2}{Mc} \left( -\delta_{mn} - \frac{\omega_q^2}{\omega_q^2 - \varepsilon^2} q_m q_n \right) B^m(q, \varepsilon) = - \frac{N_0 e^2}{Mc} B^m(q, \varepsilon)$$

(60)

The second term in the brackets of these equations vanishes because of $q_m E^m(q, \varepsilon) = \nabla E(q, \varepsilon) = 0$ and $q_n B^n(q, \varepsilon) = \nabla B(q, \varepsilon) = 0$. These equations present the generalizations of the classical London equations, Eqs. (6) and (7) illustrated in Sec. IV, which extend their range of validity beyond that of steady state homogeneous super-currents to alternating currents with a wavelength $q^{-1}$ and frequency $\varepsilon$. It permits us to illustrate the emergence of massive photons of the electromagnetic field which accompanies these currents. In order to see that let us act onto Eq. (59) once more with $i\varepsilon$ and on Eq. (60) with $q$.

We thus derive the wave equations for the electromagnetic excitations

$$\left( \varepsilon^2 - c^2 q^2 \right) \langle A(q, \varepsilon) \rangle_A = \frac{N_0 e^2}{Mc} \frac{\varepsilon^2}{\omega_q^2 - \varepsilon^2} A^m(q, \varepsilon)$$

(61)

where the last equality arises from the Ampere-Maxwell relation $q \times B(q, \varepsilon) - (i\varepsilon/c) E(q, \varepsilon) = - \langle J(q, \varepsilon) \rangle_A$. Eq. (61) describes the Proca equations for massive spin-1 vector boson electromagnetic excitations.

Projecting out in Eq. (53) the longitudinal and transverse components of the current $\langle J(q, \varepsilon) \rangle_A$,

$$J^m_{\text{long}} = \frac{1}{q^2} q_m q_n \langle J^n(q, \varepsilon) \rangle_A = \frac{N_0 e^2}{Mc} \frac{\varepsilon^2}{\omega_q^2 - \varepsilon^2} A^m_{\parallel}(q, \varepsilon)$$

(62)

$$J^m_{\text{trans}} = \frac{1}{q^2} (q_m q_n - \delta_{mn} q^2) \langle J^n(q, \varepsilon) \rangle_A = - \frac{N_0 e^2}{Mc} A^m_{\perp}(q, \varepsilon)$$

(63)

describes the response to the longitudinal and transverse components of the vector field $A(q, \varepsilon)$, parallel and orthogonal to the direction of the propagation of the electromagnetic waves, given by

$$A^m_{\parallel}(q, \varepsilon) = \frac{q_m q_n}{q^2} \langle A^n(q, \varepsilon) \rangle_A = \frac{q_m q_n - \delta_{mn} q^2}{q^2} \langle A^n(q, \varepsilon) \rangle_A = 0$$

(64)

$$A^m_{\perp}(q, \varepsilon) = \frac{q_m q_n - q^2 \delta_{mn}}{q^2} \langle A^n(q, \varepsilon) \rangle_A = \frac{q_m A^n(q, \varepsilon)}{q^2} = 0.$$

(65)

We thus derive the wave equations for the electromagnetic excitations

$$\left( \varepsilon^2 - c^2 q^2 \right) \langle A_{\parallel}(q, \varepsilon) \rangle_A = 0$$

(66)

$$\left( \varepsilon^2 - c^2 q^2 - \frac{N_0 e^2}{Mc} \right) \langle A_{\perp}(q, \varepsilon) \rangle_A = 0,$$

(67)

which locally accompany the super-current. In the long wavelength limit, the longitudinal and the transverse photons exhibit identical finite masses $m_0 = \frac{\hbar}{c} \sqrt{\frac{N_0 e^2}{Mc}}$.

From the results obtained in this section it follows that a free charged Bose liquid, exposed to an external magnetic field, exhibits a resistance-less super-current and an associated to it Meissner-Ochsenfeld magnetic field screening. The London equations, which we derived here for finite frequency and wave vector dependend magnetic fields remain the same for a weakly interacting charged Bose liquid. We simply have to replace $\omega_q$ by a corresponding linear dispersion $\propto g q$, which derives from Bogoliubov modes (see for details ref. 69) which do invoke a SSB state of a phase-polarized ground state: a macroscopic quantum coherent state with an arbitrary but fixed phase of the condensate. Given this situation that the superconducting state derived (i) on the basis of an apparently spontaneously symmetry broken SSB state, as expected for an interacting Bose liquid and (ii) on the basis of a single macroscopically occupied state,
as expected for a non-interacting bose liquid, brings us back to the old question whether the phase of an isolated superconducting condensate has a real physical meaning.

Phil Anderson considered in this context two superconductors, prepared independently and whose condensate phases are consequently arbitrary. Will this lead to a Josephson current controlled by a specific or an arbitrary phase difference when they are connected by a super-leak? A related scenario concerns a gas of atomic hydrogen prepared in two spin polarized states with different phases above their condensation temperature \( T_{\lambda} \). The question then is whether such a system, upon lowering the temperature to below \( T_{\lambda} \), condenses into two separate condensates with different phases or into a single uniquely defined phase, determined by the difference of their two phases. (See also the discussion on that in ref. [5]). The question of whether an isolated superconductor has a definite physically accessible phase or not remains so far unanswered and with it the question of whether SSB of a polarized Many Body ground state with a physically meaningful phase is a prerequisite for electromagnetic gauge symmetry breaking or not. Our example of the charged free Bose liquid seem to suggest that a macroscopically occupied single-particle state is enough for that.

### VII. SUMMARY AND OUTLOOK

The truely profound physics behind the phenomenon of superconductivity became apparent only in the years following the publication of the BCS theory, which had explained the totality of thermodynamic and transport features of these materials. We have reviewed and highlighted here the revolutionizing new concepts which we had learned from that and which were crucial for the development of the whole of physics ever since. We focused here on the major theoretical issues:

(i) Gauge fields, which accompany particle matter-waves in their scattering and decay processes by safeguarding the basic conservation laws of nature. (ii) Spontaneous breaking of continuous symmetries of quantum vacuums, which generates conserved currents which describe collective Nambu-Goldstone modes, associated to conserved quantities.

(iii) The emergence of massive excitations out of such quantum vacuums arises in a natural way through the gauge relations linking the two: The massless Nambu-Goldstone modes and the massless Gauge bosons, via the Anderson-Higgs mechanism. It ultimately stipulates the existence of a yet to be experimetaly verified Higgs boson, presenting the source field from which all mass in our universe should originate.

We owe it to the particle field theory community, for having to a large extent unearthed these concepts, which they put to great use in consolidating the quantum gauge field theoretical approach in terms of the standard model. In return, it had permitted us to regard superconductivity from a broader view, which has become invaluable in adressing its hitherto unsuspected manifestations in the high \( T_{\lambda} \) cuprates and the cold atomic gases.

Transposing the field-theoretical concepts onto our traditional solid state physics vision of superconductivity, we arrive at the following picture: (A) The gauge bosons of the quantum field theory play the role of the excitations of the electromagnetic field in which the superconductor has to be embedded in order to produce its fundamental features: the persistant resistance-less super-currents. (B) Spontaneous symmetry breaking of the highly polarizable Many Body ground states of attractively interacting electrons results in an ensemble of phase-locked Cooper pairs all around the Fermi surface, together with its collective Nambu-Goldstone phase fluctuations. (C) The Anderson-Higgs mechanism, transforming massless gauge bosons into massive ones provides in a superconductor the feedback between the circulating charge carrying current and the magnetic field which it induces. Its response back onto the current stabilizes that later in form of its steady state persistent resistanceless conduction, which is accompanied by a self-induced electromagnetic field component whose photonic excitations are massive - as manifest in Meiszer-Ochsenfeld screening. This feature presents the most profound aspect of the superconducting state: its broken electromagnetic gauge symmetry.

Although by the late sixties, all aspects of superconductivity had been understood, no significant progress had been made to find a way to increase its critical temperature, which evidently always had been the most important practical issue of it. However, in 1986 a new class of superconductors had been discovered, which nowadays can provide us with \( T_{\lambda} \)'s of around 150 \( K \). It was clear from the begining that these superconductors were not of BCS type for which \( T_{\lambda} \) is controlled by the binding energy of the collective Cooper pair state, i.e., the gap. This BCS scenario prevents \( T_{\lambda} \) to become much higher than about 30 \( K \). The new superconductors is controlled by the density to mass ratio of the superfluid carriers \( T_{\lambda} \propto \frac{n_s}{m_s} \) and that can reach sizeable values, provided that the effective bosonic charge carriers can Bose-Einstein condense into a superfluid state.

At one time, a promising idea was to consider systems with strong electron-phonon coupling which could lead to a superfluid state of bosonic bipolarons - Bipolaronic superconductivity. But unfortunately Bipolarons seem to be condemned to never exist as mobile Bloch states and prefer to condense in insulating states. According to a generally accepted point of view in the chemist's community, high \( T_{\lambda} \)'s should be possible to be achieved in strong electron-lattice coupled systems, provided they display intrinsic local dynamical lattice instabilities. This suggestion led me in
the early eighties, well before the discovery of the cuprate superconductors, to propose the phenomenological Boson-Fermion Model (BFM) for such a situation. In essence it describes itinerant electrons in a dynamically deformable background of molecular clusters, which momentarily trap electrons in form of resonating pairs, similar to Feshbach resonance pairing in atomic gases. On a local level, electrons then exist simultaneously in form of itinerant states \( c_i^\dagger = (1/N) \sum_k \exp(i \mathbf{k} \cdot \mathbf{r}_i) c_{i,k} \), hopping on and off such molecular clusters and states in which they are tightly bound together in form of localized bipolarons \((BP^1)\). This manifests itself in form of a local quantum superposition of these two configurations: of bonding and anti-bonding two-particle states \((1/\sqrt{2})[c_i^\dagger c_k^\dagger \mp BP^1]\) coexisting with itinerant single-particle states \(c_i^\dagger\). It is this feature which permits one to achieve bipolarons in phase locked coherent states, which in an underlying metallic crystalline solid, can condense into a superfluid state, without that they have to have any prior to it intrinsic itinerancy. This feature is protected from the aces of the individually locally fluctuating molecular environment (which forms them in the first place) by their being locked up in the collective excitations of the Many Body system.

The BFM scenario presents a play-ground for what is generally termed emerging phenomena\(^{43,87}\) in Many Body systems and the associated quantum protection\(^{88,89}\) which assures their dynamical stability. The laws which govern emerging phenomena derive from "macroscopic conservation laws". They are dynamically uncorrelated to the laws which control their constituents on a local spatial level and which form a highly degenerate quantum vacuum. The local constituents, having lost any individuality manifest themselves in the collective excitations of the macroscopic system, which "protects them against the vicissitude of the laws governing the individual constituents."

In the BFM, the local physics is controlled by local phase correlations between itinerant pairs of electrons passing momentarily through such local molecular sites and bound pairs of them, which forms a highly degenerate quantum vacuum, composed of such configurations. It characterizes the pseudogap state, respectively the insulating phase of the underdoped parent compound, at low temperatures, out of which emerges the superconducting state upon doping, respectively reducing the temperature\(^{84,90}\). The local phase correlations are in competition with the phase correlation controlling the macroscopic physics and which try to link together the bosonic components \(BP\) of spatially separated local molecular bonding states\(^{28}\). In this process those latter loose their well defined local spectral individuality: a characteristic Boson-Fermion Duality\(^{85}\) in form of a three peak structure in their local single-particle spectral function. As extended single-particle states are formed out of these individual cluster states, the corresponding single-particle spectral features are no longer being describeable in terms of simple pole singularities,\(^{22}\) after having been integrated into the dynamics of the collective excitations of the macroscopic system.

The physics inherent in the novel high \(T_c\) superconductors is clearly not contained in the physics we have learned from the BCS superconductors. New concepts like that of emergence and quantum protection and most likely more to come have to be acquired to eventually understand the new superconductors on the level we have understood those of BCS systems.

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1. H. Kamerling Onnes, Acad. van Wetenschappen (Amsterdam) 14, 113 (1911).


